Plan

Molecular Electronic-Structure Theory, by Trygve Helgaker, Poul Jørgensen, Jeppe Olsen

1. - CASCI §11.1
   - Orbital rotation §3
   - 2nd-order SCF §10.7–10.10
   - CASSCF §5.5, §12
2. MRPT2 §14.7
3. MRCI, DFT/MRCI

Further topics

- Spin projection
- GW/BSE

Overview

Optimization Methods: Basics

$$E(\mathbf{X}_1) = E(\mathbf{X}) + \mathbf{q}^\dagger \mathbf{f}(\mathbf{X}) + \frac{1}{2}\mathbf{q}^\dagger \mathbf{H}(\mathbf{X})\mathbf{q} + \cdots \tag{1.1}$$

$$\mathbf{f}(\mathbf{X}_1) = \mathbf{f}(\mathbf{X}) + \mathbf{H}(\mathbf{X})\mathbf{q} \tag{1.2}$$

where

$$\mathbf{q} = \mathbf{X}_1 - \mathbf{X} \qquad f_i = \frac{\partial E(\mathbf{X})}{\partial X_i} \qquad H_{ij} = \frac{\partial^2 E(\mathbf{X})}{\partial X_i \partial X_j} \tag{1.3}$$

Newton Method, aka Newton-Raphson Method

Let $\mathbf{X}_1 = \mathbf{X}_e$

$$\mathbf{f}(\mathbf{X}) = -\mathbf{H}(\mathbf{X})\mathbf{q} \tag{1.4}$$

i.e.

$$\mathbf{q} = -\mathbf{H}^{-1}(\mathbf{X})\mathbf{f}(\mathbf{X}) \tag{1.5}$$

Quadratic Convergence of Newton's Method

If f is continuously differentiable and its derivative is not 0 at α and it has a second derivative at α then the convergence is quadratic or faster.

$$\Delta x_{i+1} = \frac{f''(\alpha)}{2f'(\alpha)}(\Delta x_i)^2 + O(\Delta x_i)^3 \tag{1.6}$$

Quasi-Newton Optimization

Murtagh-Sargent Method, Szabo §C

$$\mathbf{q}_n = -\alpha_{n-1}\mathbf{G}_{n-1}\mathbf{f}_{n-1} \tag{1.7}$$

1. Set $\alpha_0 = 1$ and $\mathbf{G_0} = \mathbf{I}$. While $(E_1 > E_0)$ set $\alpha_0 \leftarrow \alpha_0/2$
2. $$\mathbf{U}_k = -\alpha_{k-1}\mathbf{G}_{k-1}\mathbf{f}_{k-1} - \mathbf{G}_{k-1}(\mathbf{f}_k - \mathbf{f}_{k-1}) \tag{1.8}$$
   $$a_k^{-1} = \mathbf{U}_k^\dagger\mathbf{d}_k = \mathbf{U}_k^\dagger(\mathbf{f}_k - \mathbf{f}_{k-1}) \tag{1.9}$$
   $$T_k = \mathbf{U}_k^\dagger\mathbf{U}_k \tag{1.10}$$
   if $a_k^{-1} < 10^{-5}T_k$ or $a_k\mathbf{U}_k^\dagger\mathbf{f}_{k-1} > 10^{-5}$, goto step 1
   else
   $$\mathbf{G}_k = \mathbf{G}_{k-1} + a_k\mathbf{U}_k\mathbf{U}_k^\dagger \tag{1.11}$$
   $$\alpha_k = 1 \tag{1.12}$$
3. $$\mathbf{q}_k = -\alpha_{k-1}\mathbf{G}_{k-1}\mathbf{f}_{k-1} \tag{1.13}$$

Quasi-Newton Optimization

Method	$B_{k+1} =$	$H_{k+1} = B_{k+1}^{-1} =$
BFGS	$B_k + \frac{y_k y_k^{\mathrm{T}}}{y_k^{\mathrm{T}} \Delta x_k} - \frac{B_k \Delta x_k (B_k \Delta x_k)^{\mathrm{T}}}{\Delta x_k^{\mathrm{T}} B_k \, \Delta x_k}$	$\left(I - \frac{\Delta x_k y_k^{\mathrm{T}}}{y_k^{\mathrm{T}} \Delta x_k}\right) H_k \left(I - \frac{y_k \Delta x_k^{\mathrm{T}}}{y_k^{\mathrm{T}} \Delta x_k}\right) + \frac{\Delta x_k \Delta x_k^{\mathrm{T}}}{y_k^{\mathrm{T}} \, \Delta x_k}$
Broyden	$B_k + \frac{y_k - B_k \Delta x_k}{\Delta x_k^{\mathrm{T}} \, \Delta x_k} \Delta x_k^{\mathrm{T}}$	$H_k + \frac{(\Delta x_k - H_k y_k) \Delta x_k^{\mathrm{T}} H_k}{\Delta x_k^{\mathrm{T}} H_k \, y_k}$
Broyden family	$(1 - \varphi_k) B_{k+1}^{\mathrm{BFGS}} + \varphi_k B_{k+1}^{\mathrm{DFP}}, \quad \varphi \in [0, 1]$	
DFP	$\left(I - \frac{y_k \, \Delta x_k^{\mathrm{T}}}{y_k^{\mathrm{T}} \, \Delta x_k}\right) B_k \left(I - \frac{\Delta x_k y_k^{\mathrm{T}}}{y_k^{\mathrm{T}} \, \Delta x_k}\right) + \frac{y_k y_k^{\mathrm{T}}}{y_k^{\mathrm{T}} \, \Delta x_k}$	$H_k + \frac{\Delta x_k \Delta x_k^{\mathrm{T}}}{\Delta x_k^{\mathrm{T}} \, y_k} - \frac{H_k y_k y_k^{\mathrm{T}} H_k}{y_k^{\mathrm{T}} H_k y_k}$
SR1	$B_k + \frac{(y_k - B_k \, \Delta x_k)(y_k - B_k \, \Delta x_k)^{\mathrm{T}}}{(y_k - B_k \, \Delta x_k)^{\mathrm{T}} \, \Delta x_k}$	$H_k + \frac{(\Delta x_k - H_k y_k)(\Delta x_k - H_k y_k)^{\mathrm{T}}}{(\Delta x_k - H_k y_k)^{\mathrm{T}} y_k}$

2nd-Order Hartree-Fock

```
            approx. NR    full NR
Gaussian:                 scf=qc
ORCA:        SOSCF        NRSCF
PySCF:                    scf.RHF().newton()
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Orbital Rotation: Basics

matrix exponential

$$\mathrm{e}^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} \tag{1.14}$$

性质

$$\mathbf{B}\,\mathrm{e}^{\mathbf{A}}\,\mathbf{B}^{-1} = \mathrm{e}^{\mathbf{BAB}^{-1}} \tag{1.15}$$

$$\mathrm{e}^{\mathbf{A}}\,\mathbf{B}\,\mathrm{e}^{-\mathbf{A}} = \mathbf{B} + [\mathbf{A}, \mathbf{B}] + \frac{1}{2!}[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] \tag{1.16}$$

exponential representation of unitary matrices

For any unitary matrix $\mathbf{U}$, we can always find an anti-Hermitian matrix $\mathbf{X}$ such that

$$\mathbf{U} = \mathrm{e}^{\mathbf{X}} \quad \mathbf{X}^{\dagger} = -\mathbf{X} \tag{1.17}$$

Proof:

$$\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{I} \Rightarrow \mathbf{X} + \mathbf{X}^{\dagger} = 0 \tag{1.18}$$

Unitary Orbital Transformation

$$\tilde{\phi}_p = \sum_q \phi_q U_{qp} \quad \text{or } \tilde{\mathbf{C}} = \mathbf{CU} \tag{1.19}$$

the unitary matrix $\mathbf{U}$ may be written in terms of an anti-Hermitian matrix $\boldsymbol{\kappa}$ as

$$\mathbf{U} = \mathrm{e}^{-\boldsymbol{\kappa}} \tag{1.20}$$

operator form

$$\mathrm{e}^{\hat{\kappa}} = \sum_{n=0}^{\infty} \frac{\hat{\kappa}^n}{n!} \quad \hat{\kappa} = \sum_{pq} \kappa_{pq} a_p^\dagger a_q \tag{1.21}$$

Exponential Parameterization of Density Matrix

MO-based

$$\boldsymbol{\rho}(\boldsymbol{\kappa}) = \mathrm{e}^{-\boldsymbol{\kappa}}\,\boldsymbol{\rho}\,\mathrm{e}^{\boldsymbol{\kappa}} = \mathrm{e}^{-\boldsymbol{\kappa}} \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix} \mathrm{e}^{\boldsymbol{\kappa}} \tag{1.22}$$

AO-based

$$\begin{aligned} \mathbf{R}(\boldsymbol{\kappa}) &= \mathbf{C}\boldsymbol{\rho}(\boldsymbol{\kappa})\mathbf{C}^T = \mathbf{C}\,\mathrm{e}^{-\boldsymbol{\kappa}}\,\boldsymbol{\rho}\,\mathrm{e}^{\boldsymbol{\kappa}}\,\mathbf{C}^T \\ \mathbf{R} &= \mathbf{C}\boldsymbol{\rho}\mathbf{C}^T \end{aligned} \tag{1.23}$$

$\therefore$

$$\begin{aligned} \mathbf{R}(\boldsymbol{\kappa}) &= \mathbf{C}\,\mathrm{e}^{-\boldsymbol{\kappa}}[\mathbf{C}^T\mathbf{S}\mathbf{C}]\boldsymbol{\rho}[\mathbf{C}\mathbf{S}\mathbf{C}^T]\,\mathrm{e}^{\boldsymbol{\kappa}}\,\mathbf{C}^T \\ &= \mathrm{e}^{-\mathbf{X}\mathbf{S}}\,\mathbf{R}\,\mathrm{e}^{\mathbf{S}\mathbf{X}} \qquad \text{with } \mathbf{X} = \mathbf{C}\boldsymbol{\kappa}\mathbf{C}^T \end{aligned} \tag{1.24}$$

BCH expansion

$$\mathbf{R}(\mathbf{X}) = \mathbf{R} + [\mathbf{R},\mathbf{X}]_S + \frac{1}{2}[[\mathbf{R},\mathbf{X}]_S,\mathbf{X}]_S \tag{1.25}$$

where $[\mathbf{R},\mathbf{X}]_S = \mathbf{RSX} - \mathbf{XSR}$

Electronic Gradient

$$E = 2\,\mathrm{tr}\,\mathbf{hR} + \mathrm{tr}\,\mathbf{RG}(\mathbf{R}) \tag{1.26}$$

HF electronic gradient

$$\begin{aligned} E(\mathbf{X}) &= E^{(0)} + 2\,\mathrm{tr}\,\mathbf{h}[\mathbf{R},\mathbf{X}]_S + 2\,\mathrm{tr}\,\mathbf{h}\frac{1}{2}[[\mathbf{R},\mathbf{X}]_S,\mathbf{X}]_S + \mathrm{tr}[\mathbf{R},\mathbf{X}]_S\mathbf{G}(\mathbf{R}) + \mathrm{tr}\,\mathbf{RG}([\mathbf{R},\mathbf{X}]_S) \\ &\quad + \frac{1}{2}\,\mathrm{tr}[[\mathbf{R},\mathbf{X}]_S,\mathbf{X}]_S\mathbf{G}(\mathbf{R}) + \frac{1}{2}\,\mathrm{tr}\,\mathbf{RG}([[\mathbf{R},\mathbf{X}]_S,\mathbf{X}]_S) + \mathrm{tr}[\mathbf{R},\mathbf{X}]_S\mathbf{G}([\mathbf{R},\mathbf{X}]_S) \\ &= E^{(0)} + 2\,\mathrm{tr}\,\mathbf{F}[\mathbf{R},\mathbf{X}]_S + \mathrm{tr}\,\mathbf{F}[[\mathbf{R},\mathbf{X}]_S,\mathbf{X}]_S + \mathrm{tr}[\mathbf{R},\mathbf{X}]_S\mathbf{G}([\mathbf{R},\mathbf{X}]_S) + \cdots \end{aligned} \tag{1.27}$$

$$E^{(1)}_{\mu\nu} = \frac{\partial}{\partial X_{\mu\nu}} 2\,\mathrm{tr}\,\mathbf{F}[\mathbf{R},\mathbf{X}]_S = 4(\mathbf{SRF} - \mathbf{FRS})_{\mu\nu} \tag{1.28}$$